FUNDAMENTAL GROUP IN NONZERO CHARACTERISTIC

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ABSTRACT. A proof of freeness of the commutator subgroup of the fundamental group of a smooth irreducible affine curve over a countable algebraically closed field of nonzero characteristic. A description of the abelianizations of the fundamental groups of affine curves over an algebraically closed field of nonzero characteristic is also given.

1. INTRODUCTION

The algebraic fundamental group of smooth curves over an algebraically closed field of characteristic zero is a well understood object, thanks to Grothendieck's Riemann existence theorem [SGAI, XIII, Corollary 2.12, page 392] . But if the characteristic of the base field is $p > 0$ and the curve is affine then there may be wild ramification so computing the algebraic fundamental group is not as simple. Though Grothendieck's theorem gives a description of the prime-to-p part of the fundamental group, which is analogous to the characteristic zero case. But the structure of the whole group is still elusive in spite of the fact that all the finite quotients of this group are now known. A necessary and sufficient condition for a group to be a finite quotient of the fundamental group was conjectured by Abhyankar (see the theorem below) and was proved by Raynaud [Ray] (in the case of the affine line) and Harbater [Ha1] (for arbitrary smooth affine curves). For a finite group G and a prime number p, let $p(G)$ denote the subgroup of G generated by all the p-Sylow subgroups. $p(G)$ is called the *quasi-p* part of G.

Theorem 1.1. (**Raynaud, Harbater**) *Let C be a smooth projective curve of genus g over an algebraically closed field of characteristic $p > 0$ and for some $n \geq 0$, let $x_0, .., x_n$ be some points on C. Then a finite group G is a quotient of the fundamental group $\pi_1(C \setminus \{x_0, .., x_n\})$ if and only if $G/p(G)$ is generated by $2g + n$ elements. In particular a finite group G is a quotient of $\pi_1(\mathbb{A}^1)$ if and only if $G = p(G)$, i.e., G is a quasi-p group.*

The "if part" of the above theorem is the nontrivial part, the "only if part" was proved long back by Grothendieck.

From now on we shall assume that the characteristic of the base field is $p > 0$. Consider the following exact sequence for the fundamental group of a smooth affine curve C.

$$1 \to \pi_1^c(C) \to \pi_1(C) \to \pi_1^{ab}(C) \to 1$$

where $\pi_1^c(C)$ and $\pi_1^{ab}(C)$ are the commutator subgroup and the abelianization of the fundamental group $\pi_1(C)$ of C, respectively. In this paper we give a description of the abelianization and show that the commutator subgroup is a free profinite group. The result on the commutator subgroup falls into the league of the so called

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Shafarevich conjecture for global fields. Recall that the Shafarevich conjecture says that the commutator subgroup of the absolute Galois group of the rationals $\mathbb{Q}$ is free. David Harbater ([Ha7]), Florian Pop ([Pop]) and later Dan Haran and Moshe Jarden ([HJ]) have shown, using different patching methods, that the absolute Galois group of the function field of a curve over an algebraically closed field is free. More Shafarevich conjecture type results have been proved in [Ha8].

The second section of this paper covers definitions and notations. In the third section we give a description of the p-part of the abelianization of the algebraic fundamental group of any normal affine algebraic variety over an algebraically closed field in terms of Witt vectors. We deduce the fact that the abelianization of the algebraic fundamental group determines $W_n(A)/P(W_n(A))$ as a group (see Corollary 3.6) where $W_n(A)$ is the ring of finite Witt vectors over the coordinate ring A of the affine curve under consideration and P is the additive group endomorphism of $W_n(A)$ which sends $(a_1,..,a_n)$ to $(a_1^p,..,a_n^p)-(a_1,..,a_n)$ (here $''-''$ is subtraction in the Witt ring). It is conjectured by Harbater that the algebraic fundamental group should determine A as a ring.

The rest of the paper is devoted to proving that the commutator subgroup of the fundamental group of a smooth irreducible affine curve over a countable algebraically closed field k is a free profinite group of countable rank. The fourth section consists of some group theory results and a result on projectivity of the commutator subgroup. These results allow us to reduce the problem to finding proper solution for any split embedding problem with perfect quasi-p group as the kernel, abelian p-group as kernel and prime-to-p group as kernel (for definitions see Section 1).

The fifth section is on finding solutions for all quasi-p perfect embedding problems and abelian p-group embedding problems. This is relatively simple.

The sixth section is the longest section and is devoted to finding proper solutions for prime-to-p embedding problems. Methods on formal patching developed by Harbater (see [Ha1] and [Ha2]) and their mild generalizations have been used in this section to prove the desired result.

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2. Definitions and notations

An embedding problem consists of surjections, $\phi : \pi \to G$ and $\alpha : \Gamma \to G$

$$\begin{tikzcd}
& & & \pi \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\alpha"'] & G \arrow[r] \arrow[d] & 1 \\
& & & 1 &
\end{tikzcd}$$

where G, H, Γ and π are groups and $H = \ker(\alpha)$. It is said to have a *weak solution* if there exists a group homomorphism ψ which makes the diagram commutative,

i.e., $\alpha \circ \psi = \phi$. Moreover, if ψ is an epimorphism then it is said to have a *proper* solution. It is said to be a finite embedding problem if Γ is finite. All the embedding problems considered here will be assumed to be finite. It is said to be a split embedding problem if there exists a group homomorphism from G to Γ which is a right inverse of α. It is called a quasi-p embedding problem if H is a quasi-p group, i.e., H is generated by its Sylow p-subgroups and similarly it is called a prime-to-p embedding problem if H is a prime-to-p group, i.e., order of H is prime to p. H will sometimes be referred to as the kernel of the embedding problem.

A profinite group is called *free* if it is a profinite completion of a free group. A *generating set* of a profinite group π is a subset I so that the closure of the group generated by I is the whole group π. The *rank* of a profinite group is the minimum of the cardinality of a generating set.

For a ring R, let $\mathrm{frac}(R)$ denote the total ring of quotients of R. A ring extension $R \subset S$ is said to be *generically separable* if R is a domain, $\mathrm{frac}(S)$ is separable extension of $\mathrm{frac}(R)$ and no nonzero element of R becomes a zero divisor in S.

As in [Ha2], a morphism of schemes, $\Phi : Y \to X$, is said to be a *cover* if Φ is finite and generically separable, i.e., X can be covered by affine open subset $U = \mathrm{Spec}(R)$ such that the ring extension $R \subset \mathcal{O}(\Phi^{-1}(U))$ is generically separable. For a finite group G, Φ is said to be G-*cover* if in addition there exists a group homomorphism $G \to \mathrm{Aut}_X(Y)$ which acts transitively on the geometric generic fibers of Φ.

For an integral scheme X, $\pi_1(X)$ will denote the algebraic fundamental group of X with respect to the generic point. For an affine variety X over a field k, $k[X]$ will denote the coordinate ring of X and $k(X)$ will denote the function field. For a scheme X and a point $x \in X$, let $\hat{\mathcal{K}}_{X,x}$ denote the fraction field of complete local ring $\hat{\mathcal{O}}_{X,x}$ whenever the latter is a domain. For domains $A \subset B$, $\overline{A}^B$ will denote the integral closure of A in B.

For a scheme X, let $\mathcal{M}(X)$ denote the category of coherent sheaves of $\mathcal{O}_X$-modules, $\mathcal{AM}(X)$ denote the category of coherent sheaves of $\mathcal{O}_X$-algebras and $\mathcal{SM}(X)$ denote the subcategory of $\mathcal{AM}(X)$ for which the sheaves are generically separable and locally free. For a finite group G, let $G\mathcal{M}(X)$ denote the category of generically separable coherent locally free sheaves of $\mathcal{O}_X$-algebras S together with a G-action which is transitive on the geometric generic fibers of $\mathrm{Spec}_{\mathcal{O}_X}(S) \to X$. For a ring R, we may use $\mathcal{M}(R)$ instead of $\mathcal{M}(\mathrm{Spec}(R))$, etc. Given categories $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ and functors from $\mathcal{F} : \mathcal{A} \to \mathcal{C}$ and $\mathcal{G} : \mathcal{B} \to \mathcal{C}$, we define the fiber product category $\mathcal{A} \times_{\mathcal{C}} \mathcal{B}$ whose objects are triples (A, B, C), where A, B and C are objects of $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ respectively together with isomorphisms of C with $\mathcal{F}(A)$ and with $\mathcal{G}(B)$ in $\mathcal{C}$, morphisms are triples (a, b, c), where a, b and c are morphisms in $\mathcal{A}$, $\mathcal{B}$ and $\mathcal{C}$ respectively, so that $\mathcal{F}(a)$ and $\mathcal{G}(b)$ under the functors $\mathcal{F}$ and $\mathcal{G}$ are morphism in $\mathcal{C}$ which agrees with c in the natural way. That is, the following two squares commute.

$$\begin{tikzcd} C \arrow[r, "c"] \arrow[d] & C' \arrow[d] \\ \mathcal{F}(A) \arrow[r, "\mathcal{F}(a)"'] & \mathcal{F}(A') \end{tikzcd} \qquad \begin{tikzcd} C \arrow[r, "c"] \arrow[d] & C' \arrow[d] \\ \mathcal{G}(B) \arrow[r, "\mathcal{G}(b)"'] & \mathcal{G}(B') \end{tikzcd}$$

3. ABELIANIZATION

Let $X = \mathrm{Spec}(A)$ be a normal affine algebraic variety over an algebraically closed field k of characteristic $p > 0$. For a ring R, by $(W_n(R), +, .)$ we denote the ring of Witt vectors of length n over R. Let F be the Frobenius map on $W_n(R)$ which sends $(a_1, .., a_n)$ to $(a_1^p, .., a_n^p)$ and $P : W_n(R) \to W_n(R)$ be the map of abelian groups sending $(a_1, .., a_n) \mapsto (a_1^p, .., a_n^p) - (a_1, ..a_n)$, i.e., F - Identity. For a detailed account of Witt vectors, readers are advised to see [Jac, Chapter 8]. In fact, some of the ideas for the proof of Lemma 3.3 below come from this source.

Let $G = \pi_1^{ab}(X)$ and let G_p be the maximal p quotient of G, i.e., G_p is the quotient group of G such that every finite p-group quotient of G factors through G_p. Note that G_p is also a subgroup of G. Let $G_W = \mathrm{Hom}(\varinjlim W_n(A)/P(W_n(A)), S^1)$, where the group homomorphism from $W_n(A)/P(W_n(A))$ to $W_{n+1}(A)/P(W_{n+1}(A))$ is given by $[(a_1, .., a_n)] \mapsto [(0, a_1, ..a_n)]$.

Theorem 3.1. *The p-part of the abelianization of the fundamental group G_p is isomorphic to G_W.*

First we need a few lemmas. For a sheaf of rings $\mathcal{F}$ of characteristic p on a topological space X, let $W_n(\mathcal{F})$ denote the sheaf which assigns to an open set U, the ring $W_n(\mathcal{F}(U))$. A version of the following lemma can be found in [Se1].

Lemma 3.2. *(***Serre***) Let A be a noetherian ring of characteristic p. Let B, also of characteristic p, be a finite ring extension of A then for every $n \geq 1$, $H^k_{et}(\mathrm{Spec}(A), W_n(\mathcal{B})) = 0$, $k > 0$, where $\mathcal{B} = \theta_* \mathcal{O}_{\mathrm{Spec}(B)}$ and $\theta : \mathrm{Spec}(B) \to spec(A)$ is the morphism induced from $A \hookrightarrow B$.*

Proof. B is a finite module over A, hence $\mathcal{B}$ is coherent sheaf over $\mathrm{Spec}(A)$. We shall use induction on n to prove the lemma. Note that $W_1(\mathcal{B}) = \mathcal{B}$. By the Serre's vanishing theorem and the fact that étale cohomology of coherent sheaves agrees with the Zariski cohomology (see [Mil, 3.7, 3.8, page 114]), the lemma holds for $n = 1$. For the induction step, consider the following exact sequence.

$$0 \longrightarrow \mathcal{B} \longrightarrow W_{n+1}(\mathcal{B}) \longrightarrow W_n(\mathcal{B}) \longrightarrow 0$$

where for a fixed open set U the surjection (on the level of rings) is given by $(b_1, .., b_n, b_{n+1}) \mapsto (b_1, .., b_n)$, clearly the kernel is $\{(0, .., 0, b) \in W_{n+1}(\mathcal{B}(U)) : b \in \mathcal{B}(U)\} \cong \mathcal{B}(U)$. This induces a long exact sequence

$$... \to H^k_{et}(\mathrm{Spec}(A), \mathcal{B}) \longrightarrow H^k_{et}(\mathrm{Spec}(A), W_{n+1}(\mathcal{B})) \to H^k_{et}(\mathrm{Spec}(A), W_n(\mathcal{B})) \longrightarrow ...$$

By the induction hypothesis on n, $H^k_{et}(\mathrm{Spec}(A), W_n(\mathcal{B})) = 0$ for $k > 0$ and hence $H^k_{et}(\mathrm{Spec}(A), W_{n+1}(\mathcal{B})) = 0$, for all $k > 0$. $\square$

Now extending the Artin-Schrier theory to the Witt vectors we get the following result.

Lemma 3.3. *Let A be a finitely generated normal domain over an algebraically closed field k of characteristic $p > 0$. Let $\pi_1(X)$ be the fundamental group of $X = \mathrm{Spec}(A)$ and $(W_n(A), +, .)$ be the ring of Witt vectors of length n. Let P be the additive group endomorphism, $F - Id$, of $W_n(A)$. Then for every $n \geq 1$, we have a natural isomorphism $W_n(A)/P(W_n(A)) \xrightarrow{\Phi} \mathrm{Hom}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$ so that the following diagram commutes.*

$$\begin{tikzcd}
W_n(A)/P(W_n(A)) \arrow[r] \arrow[d] \arrow[dr, phantom, "\#"] & \mathrm{Hom}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z})) \arrow[d] \\
W_{n+1}(A)/P(W_{n+1}(A)) \arrow[r] & \mathrm{Hom}(\pi_1(X), W_{n+1}(\mathbb{Z}/p\mathbb{Z}))
\end{tikzcd}$$

where the first vertical map sends $[(a_1, .., a_n)]$ to $[(0, a_1, .., a_n)]$ and second vertical map is induced by inclusion of $W_n(\mathbb{Z}/p\mathbb{Z})$ in $W_{n+1}(\mathbb{Z}/p\mathbb{Z})$

Proof. Let K be an algebraic closure of the fraction field $\mathrm{frac}(A)$ of A. Let K^{un} be the compositum of all subfields of K, L, so that $L/\mathrm{frac}(A)$ is finite field extension and the integral closure in L, $\overline{A}^L$ is an unramified (hence étale) ring extension of A. We define a map ϕ from $W_n(A) \to \mathrm{Hom}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$. Given $(a_1, .., a_n) \in W_n(A)$, let $(r_1, .., r_n) \in W_n(K)$ be such that $P(r_1, .., r_n) = (a_1, .., a_n)$. Note that $r_1, .., r_n \in K^{un}$; to see this first observe that $r_i^p - r_i \in A[r_1, .., r_{i-1}]$ for each $i \geq 1$. We know that the ring extension given by the polynomial of the form $Z^p - Z - a$, for $a \in A$ is unramified over A. $\pi_1(X) = \mathrm{Gal}(K^{un}/\mathrm{frac}(A))$, so $\pi_1(X)$ acts on K^{un} fixing $\mathrm{frac}(A)$. For $g \in \pi_1(X)$, let $\phi(a_1, .., a_n)(g) = (gr_1, .., gr_n) - (r_1, .., r_n)$. Note that

$$\begin{aligned}
P((gr_1, .., gr_n) - (r_1, .., r_n)) &= P((gr_1, .., gr_n)) - P(r_1, .., r_n) \\
&= gP(r_1, .., r_n) - (a_1, .., a_n) = 0.
\end{aligned}$$

Since $(r_1^p, .., r_n^p) - (r_1, .., r_n)$ is given by polynomial in $r_1, .., r_n$ with integer coefficients. Hence $F((gr_1, .., gr_n) - (r_1, .., r_n)) = (gr_1, .., gr_n) - (r_1, .., r_n)$, which gives us $(gr_1, .., gr_n) - (r_1, .., r_n) \in W_n(\mathbb{Z}/p\mathbb{Z})$. To see that $\phi(a_1, .., a_n)$ is independent of the choice of $(r_1, .., r_n)$, let $(s_1, .., s_n)$ be such that $P(s_1, .., s_n) = (a_1, .., a_n)$ then $(r_1, .., r_n) - (s_1, .., s_n) \in W_n(\mathbb{Z}/p\mathbb{Z})$, hence fixed by g. So $g((r_1, .., r_n) - (s_1, .., s_n)) = (r_1, .., r_n) - (s_1, .., s_n)$ which yields $(gr_1, .., gr_n) - (r_1, .., r_n) = (gs_1, ..gs_n) - (s_1, ..s_n)$. Next we shall see $\phi(a_1, .., a_n)$ is a homomorphism from $\pi_1(X)$ to $W_n(\mathbb{Z}/p\mathbb{Z})$. Let $g, h \in \pi_1(X)$ then

$$\begin{aligned}
\phi(a_1, .., a_n)(gh) &= (ghr_1, .., ghr_n) - (r_1, .., r_n) \\
&= (ghr_1, .., ghr_n) - (hr_1, .., hr_n) + (hr_1, .., hr_n) - (r_1, ..r_n) \\
&= \phi(a_1, .., a_n)(g) + \phi(a_1, .., a_n)(h)
\end{aligned}$$

since $P(hr_1, .., hr_n) = (a_1, .., a_n)$. Now we shall see ϕ is a homomorphism. To simplify notation, we may write $\underline{a}$ for $(a_1, .., a_n)$. Let $\underline{a}, \underline{b} \in W_n(A)$ and $\underline{r}, \underline{s} \in W_n(K)$ be such that $P(\underline{r}) = \underline{a}, P(\underline{s}) = \underline{b}$ then $P(\underline{r} + \underline{s}) = \underline{a} + \underline{b}$. Hence $\phi(\underline{a} + \underline{b}) = \phi(\underline{a}) + \phi(\underline{b})$. To determine the kernel of ϕ, note that $\phi(\underline{a}) = 0$ iff $g\underline{r} = \underline{r}$ for all $g \in G$, i.e. $\underline{r} \in W_n(\mathrm{frac}(A))$ but A is normal, hence $\underline{r} \in W_n(A)$. Hence the kernel of ϕ is $P(W_n(A))$. Let Φ be the induced map on $W_n(A)/P(W_n(A))$. The fact that the diagram commutes is obvious by the construction. So to complete the proof it suffices to show that ϕ is surjective.

Let $\alpha \in \mathrm{Hom}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$, we shall find a Witt vector $(a_1, .., a_n)$ so that $\alpha = \phi(a_1, .., a_n)$. Note that α corresponds to a Galois étale extension B of A with Galois group of $\mathrm{frac}(B)$ over $\mathrm{frac}(A)$ being $\mathrm{im}(\alpha)(= H$ say). Let $SW_n(B) = \{(r_1, .., r_n) \in W_n(B) : P(r_1, .., r_n) \in W_n(A)\}$. Clearly $W_n(A) \hookrightarrow SW_n(B)$. Let $\hat{H} = Hom(H, S^1)$ be the character group of H. For $\underline{r} \in SW_n(B)$ and $h \in H$ define

$\chi_{\underline{r}}(h) = h\underline{r} - \underline{r} := g\underline{r} - \underline{r}$ where g is any element of $\alpha^{-1}(h)$. As noted earlier $\chi_{\underline{r}}$ is a character (after identifying $W_n(\mathbb{Z}/p\mathbb{Z})$ with the unique cyclic subgroup of S^1). We know that $\hat{H} \cong H$. Also as seen earlier $\lambda : \underline{r} \mapsto \chi_{\underline{r}}$ is a group homomorphism from $SW_n(B)$ to $\hat{H}$ whose kernel is precisely $W_n(A)$. So we have an exact sequence

$$0 \longrightarrow W_n(A) \longrightarrow SW_n(B) \longrightarrow \hat{H}$$

where the last homomorphism is λ. Next we shall show that λ is surjective. Since $\hat{H} \cong H$ is a quotient of $\pi_1(X)$, we have $H^1(\hat{H}, W_n(B)) \hookrightarrow H^1(\pi_1(X), W_n(B))$ [Wei, 6.8.3]. By the Hochschild-Serre spectral sequence $H^1(\pi_1(X), W_n(B)) = H^1(\pi_1(X), H^0_{et}(X, W_n(\mathcal{B})))$ which embeds into $H^1_{et}(X, W_n(\mathcal{B}))$ (see [Mil, 2.21(b), page 106]) and by previous lemma, $H^1_{et}(X, W_n(\mathcal{B})) = 0$. So $H^1(\hat{H}, W_n(B)) = 0$, i.e., every cocycle is a coboundary. And if $\chi \in \hat{H}$ then $\chi(hh') = \chi(h) + \chi(h') = h\chi(h') + \chi(h)$, i.e., it is a cocycle hence a coboundary. So there exists $\underline{r} \in W_n(B)$ such that $\chi(h) = h\underline{r} - \underline{r}$. Since $\chi(h) \in W_n(\mathbb{Z}/p\mathbb{Z})$, $P(\chi(h)) = 0$, $\forall h \in H$, i.e. $hP(\underline{r}) = P(\underline{r}), \forall h \in H$. Since A is normal this means $P(\underline{r}) \in W_n(A)$, hence $\underline{r} \in SW_n(B)$. This proves λ is surjective. So we have $SW_n(B)/W_n(A) \cong \hat{H} \cong H$. Since H is a subgroup of $W_n(\mathbb{Z}/p\mathbb{Z})$, H has a generator of the type $h = (0, ..0, 1, 0, .., 0)$. Let the coset $(r_1, .., r_n) + W_n(A)$ be a generator of $SW_n(B)/W_n(A)$. It follows that $\chi_{\underline{r}}$ is a generator of $\hat{H}$ and hence $h_1 := \chi_{\underline{r}}(h)$ is a generator of H. So there is an $h_2 \in W_n(\mathbb{Z}/p\mathbb{Z})$ such that $h_1.h_2 = h$. Let $g \in \alpha^{-1}(h)$ then $g\underline{r} - \underline{r} = \chi_{\underline{r}}(h) = h_1$. Let $(a_1, .., a_n) = P(h_2.\underline{r})$. This Witt vector is our candidate for preimage of α, we shall show $\alpha = \phi(a_1, .., a_n)$. By assumption $\alpha(g) = h$ and

$$\begin{aligned}\phi(a_1, .., a_n)(g) &= g(h_2.\underline{r}) - h_2.\underline{r} \\ &= gh_2.g\underline{r} - h_2.\underline{r} \\ &= h_2.g\underline{r} - h_2.\underline{r} \\ &= h_2.(g\underline{r} - \underline{r}) \\ &= h_2.h_1 = h\end{aligned}$$

For arbitrary $g_1 \in G$, $\alpha(g_1) = h + .. + h$ say k times, since H is cyclic. Then

$$\begin{aligned}\phi(a_1, .., a_n)(g_1) &= h_2.(g_1\underline{r} - \underline{r}) \\ &= h_2.(\alpha(g_1)\underline{r} - \underline{r}) \\ &= h_2.(\chi_{\underline{r}}(h + .. + h)) \\ &= h_2.(\chi_{\underline{r}}(h) + .. + \chi_{\underline{r}}(h)) \\ &= h_2.(h_1 + .. + h_1) = h + .. + h\end{aligned}$$

So $\phi(a_1, .., a_n)$ agrees with α on whole of $\pi_1(X)$. $\square$

Proof. **(Theorem 3.1)** We know that $G_p \cong \mathrm{Hom}(\mathrm{Hom}_{cont}(G_p, S^1), S^1)$ by Pontriagan duality. Let K_n be the compositum of function fields of finite Galois étale extension of $\mathrm{Spec}(A)$ with Galois group a subgroup of $(\mathbb{Z}/p^n\mathbb{Z})^m$ for some m. And let G_n be the Galois group of K_n over $\mathrm{frac}(A)$. The natural group homomorphism from G_{n+1} to G_n corresponding to the Galois extension $K_{n+1} \supset K_n \supset \mathrm{frac}(A)$ makes $(G_n)_{n\geq 1}$ into an inverse system and $G_p = \varprojlim G_n$. So we have $\mathrm{Hom}_{cont}(G_p, S^1) \cong \mathrm{Hom}_{cont}(\varprojlim G_n, S^1)$. Since Hom is contravariant and dual of inverse limit is direct limit, this is isomorphic to $\varinjlim \mathrm{Hom}_{cont}(G_n, S^1)$ Now since

G_n is a p^n torsion group and $W_n(\mathbb{Z}/p\mathbb{Z})$ and be identified as the unique cyclic subgroup of S^1 of order p^n, we get $\mathrm{Hom}_{cont}(G_n, S^1) \cong \mathrm{Hom}_{cont}(G_n, W_n(\mathbb{Z}/p\mathbb{Z}))$. Also $\mathrm{Hom}_{cont}(G_n, W_n(\mathbb{Z}/p\mathbb{Z})) \cong \mathrm{Hom}_{cont}(G_p, W_n(\mathbb{Z}/p\mathbb{Z}))$, since all maps from G_p to $W_n(\mathbb{Z}/p\mathbb{Z})$ factors through G_n. Similarly, $\mathrm{Hom}_{cont}(G_p, W_n(\mathbb{Z}/p\mathbb{Z}))$ is isomorphic to $\mathrm{Hom}_{cont}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$, since $W_n(\mathbb{Z}/p\mathbb{Z})$ is an abelian p-group, all homomorphisms from $\pi_1(X)$ to $W_n(\mathbb{Z}/p\mathbb{Z})$ factors through G_p. Now by the previous lemma $\varinjlim \mathrm{Hom}_{cont}(\pi_1(X), W_n(\mathbb{Z}/p\mathbb{Z}))$ is isomorphic to $\varinjlim W_n(A)/P(W_n(A))$. □

The following result is a corollary of a classical result of Grothendieck [SGAI, XIII, Corollary 2.12, page 392].

Theorem 3.4. *(**Grothendieck**) The prime to p part of the abelianization of the fundamental group of an affine curve $C = \mathrm{Spec}(A)$ over an algebraically closed field k of characteristic $p > 0$ is given by $\bigoplus_{i=1}^{2g+r-1} (\bigoplus_{l \neq p \ prime} \mathbb{Z}_l)$ where g is the genus of the smooth compactification curve and r is the number of points in the compatification which are not in C.*

Corollary 3.5. *Under the assumption of the previous theorem, abelianization of the fundamental group of C, $\pi_1^{ab}(C)$, is given by*

$$\mathrm{Hom}(\varinjlim W_n(A)/P(W_n(A)), S^1) \bigoplus \bigoplus_{i=1}^{2g+r-1} (\bigoplus_{l \neq p \ prime} \mathbb{Z}_l)$$

Proof. This follows directly from Theorem 3.1 and Theorem 3.4.

Corollary 3.6. *Since the rank of $\pi_1^{ab}(C)$ is same as the cardinality of k, we get another proof of a known result that $\pi_1^{ab}(C)$ determines the cardinality of the base field. In fact just the p-part determines $W_n(A)/P(W_n(A))$ for all n.*

Proof. This is a direct consequence of Lemma 2.3.

4. Group theory

Let k be an algebraically closed field of characteristic $p > 0$. Let $\pi_1(C)$ be the algebraic fundamental group of a smooth affine curve C over k and $\pi_1^c(C) = [\pi_1(C), \pi_1(C)]$ be the commutator subgroup.

Theorem 4.1. *(Main theorem) Let C be an irreducible smooth affine curve over a countable algebraically closed field of characteristic p. Then $\pi_1^c(C)$ is free of countable rank.*

This will be reduced to finding solutions of certain kinds of embedding problems. Before that we need a group theoretic result which connects "freeness" of a profinite group to solving embedding problems. Below are certain results in this direction.

Theorem 4.2. *(**Iwasawa [Iwa, p.567], [FJ, Corollary 24.2]**) A profinite group π of countably infinite rank is free if and only if every finite embedding problem for π has a proper solution.*

This was generalised by Melnikov and Chatzidakis for any cardinality (cf [Jar, Theorem 2.1]). The Melnikov-Chatzidakis result says that for an infinite cardinal m, a profinite group π is free of rank m if and only if every finite nontrivial embedding problem for π has exactly m solution. Following is a variant of this result which has been proved in [HS].

Theorem 4.3. *([HS, Theorem 2.1]) Let π be a profinite group and let m be an infinite cardinal. Then π is a free profinite group of rank m if and only if the following conditions are satisfied:*
(i) π is projective.
(ii) Every split embedding problem for π has exactly m solution.

We shall anyway see a standard argument which reduces the problem of finding proper solutions of an embedding problem for a projective profinite group to finding proper solutions of a *split* embedding problem with the same kernel.

Let C be a smooth affine curve over an algebraically closed field k of cardinality m. Since $k(C)$, the function field of C, is also of cardinality m, there are only m polynomials over $k(C)$. Hence the absolute Galois group of $k(C)$ is the inverse limit of finite groups over a set of cardinality m and hence has generating set of cardinality m (generating set in the topological sense). So $\pi_1(C)$, being a quotient of the absolute Galois group of $k(C)$, is m genrated and hence $\pi_1^c(C)$ is m generated. So to prove that $\pi_1^c(C)$ is free, it suffices to show that every embedding problem for $\pi_1^c(C)$ has m proper solution (and just one solution if m is the countable cardinal), since this implies $\pi_1^c(C)$ has rank exactly m. So given an embedding problem:

$$(*)\qquad \begin{tikzcd} & & & \pi_1^c(C) \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\alpha"'] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

we need to find card$(k) = m$ proper solution (and just one solution if m is the countable cardinal) for every finite group G, Γ and H.

Before that, we shall show that $\pi_1^c(C)$ is projective and use it to reduce to the case where $(*)$ splits.

Proposition 4.4. *For an irreducible smooth affine curve C over k, $\pi_1^c(C) = [\pi_1(C), \pi_1(C)]$ is a projective group. More explicitly, given*

$$\begin{tikzcd} & \pi_1^c(C) \arrow[dl, dashed, "\exists\psi"'] \arrow[d, "\phi"] \\ \Gamma \arrow[r, two heads, "\alpha"'] & G \end{tikzcd}$$

surjections ϕ and α to a finite group G from $\pi_1^c(C)$ and another finite group Γ respectively, there exist a group homomorphism ψ from $\pi_1^c(C)$ to Γ so that the above diagram commutes, i.e., $\alpha \circ \psi = \phi$

Proof. Let K^{ab} be the compositum of the function fields of abelian étale covers of C, i.e., the compositum of all L, $k(C) \subset L \subset K^{ab}$ with $L/k(C)$ finite, the integral closure of $k[C]$ in L, $\overline{k[C]}^L$, is étale extension of $k[C]$ and $\mathrm{Gal}(L/k(C))$ is abelian.

A surjection $\phi : \pi_1^c(C) \to G$ corresponds to a Galois field extension M/K^{ab} with the Galois group $\mathrm{Gal}(M/K^{ab}) = G$ and $M \subset K^{un}$, where K^{un} is the compositum of the function fields of all étale covers of C.

Since M/K^{ab} is a finite field extension, there exist L, $k(C) \subset L \subset K^{ab}$, a finite Galois extension of $k(C)$ and L', a Galois extension of L with $\mathrm{Gal}(L'/L) = G$ and $L'K^{ab} = M$. Let $\pi_1^L = \pi_1(\mathrm{Spec}(\overline{k[C]}^L))$.

So we have the following tower of fields.

$$\begin{tikzcd}
K^{un} & M = L'K^{ab} \arrow[l] \\
K^{ab} \arrow[u, "\pi_1^c(C)"] \arrow[ur, "G"'] & L' \arrow[u] \\
L \arrow[u] \arrow[ur, "G"'] \arrow[uu, bend left=60, "\pi_1^L"] & \\
k(C) \arrow[u] &
\end{tikzcd}$$

Fig. 1

Moreover $\mathrm{Gal}(K^{un}/K^{ab}) = \pi_1^c(C)$, $\mathrm{Gal}(K^{un}/L) = \pi_1^L$ and $\pi_1^c(C)$ is a subgroup of π_1^L. The field extension L'/L gives a surjection $\tilde{\phi} : \pi_1^L \to G$. Since L'/L is a descent of the field extension M/K^{ab}, $\tilde{\phi}|_{\pi_1^c(C)} = \phi$. By [Se2, Proposition 1], which says the fundamental group of any affine curve is projective, we have $\pi_1^L := \pi_1(\mathrm{Spec}(k\overline{[C]}^L))$ is projective. So there exists a lift, $\tilde{\psi}$, to Γ of $\tilde{\phi}$. i.e.,

$$\begin{tikzcd}
 & \pi_1^L \arrow[dl, dashed, "\exists\tilde{\psi}"'] \arrow[d, "\tilde{\phi}"] \\
\Gamma \arrow[r, two heads, "\alpha"'] & G
\end{tikzcd}$$

with $\alpha \circ \tilde{\psi} = \tilde{\phi}$. So $\alpha \circ \tilde{\psi}|_{\pi_1^c(C)} = \tilde{\phi}|_{\pi_1^c(C)} = \phi$. So $\tilde{\psi}|_{\pi_1^c(C)}$ gives a lift of ϕ. $\square$

To reduce to the case where $(*)$ splits, let $G' = \mathrm{Im}\,\psi$, where ψ is as in Propostion 4.3. G' acts on H by conjugation, since H is a normal subgroup of Γ. Let $\Gamma' = H \rtimes G'$ then we have a natural surjection $\beta : \Gamma' \to \Gamma$ given by $(h, g) \mapsto hg$. So if we have a proper solution θ' for the embedding problem,

$$\begin{tikzcd}
 & & & \pi_1^c(C) \arrow[dl, dashed, "\theta'"'] \arrow[d, "\tilde{\psi}"] & \\
1 \arrow[r] & H \arrow[r] & \Gamma' \arrow[r] & G' \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd}$$

then $\theta = \beta \circ \theta'$ provides a proper solution to $(*)$. Note that this reduction holds for any projective profinite group not necessarily $\pi_1^c(C)$.

From now onwards we shall assume that all our embedding problems are split embedding problems unless otherwise stated. Our proof inducts on the cardinality of H. So whenever we encounter an embedding problem which may not be split, without explicitly stating we shall assume that the embedding problem has been replaced by an appropriate split embedding problem with the same kernel.

Theorem 4.5. *Let π be any projective profinite group of rank exactly m, then π is free of rank m if and only if for any finite group Γ and any minimal normal sugroup*

H of Γ, the embedding problem

$$\begin{tikzcd} & & & \pi \arrow[dl, dashed, "\psi"'] \arrow[d, "\phi"] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r, "\alpha"'] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

has m distinct solutions (and atleast one solution if m is the countable cardinal) in the following three cases:
(1) *H is a quasi-p perfect group, i.e. $H = [H, H]$.*
(2) *H is an abelian p-group.*
(3) *H is a prime-to-p group.*

Proof. In view of Theorem 4.3 (and Theorem 4.2 if m is the countable cardinal), "only if part" is trivial and for "if part" it is enough to show the embedding problem for π has m distinct proper solutions any finite group H. We induct on the cardinality of H. Suppose H is not minimal normal subgroup. Let H_1 be a proper nontrivial subgroup of H and H_1 is a normal subgroup of Γ. Then we have the following two proper nontrivial embeddding problems.

$$\begin{tikzcd} & & & \pi \arrow[dl, dashed] \arrow[d] & \\ 1 \arrow[r] & H/H_1 \arrow[r] & \Gamma/H_1 \arrow[r] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

and

$$\begin{tikzcd} & & & \pi \arrow[dl, dashed] \arrow[d] & \\ 1 \arrow[r] & H_1 \arrow[r] & \Gamma \arrow[r] & \Gamma/H_1 \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

Since the cardinality of H_1 and H/H_1 is strictly smaller than the cardinality of H, by induction hypothesis, we have m distinct proper solutions to these embedding problems. Hence we have m distinct proper solutions to the embedding problem. Hence we may assume H is a nontrivial minimal normal subgroup of Γ. So $H \cong \mathbb{S} \times .. \times \mathbb{S}$ for some finite simple group $\mathbb{S}$. If $\mathbb{S}$ is prime-to-p then H is prime-to-p, hence we are done by case (3). If $\mathbb{S}$ is quasi-p nonabelian group then H being product of perfect groups is perfect. So we are done by case (1). And finally if $\mathbb{S}$ is quasi-p abelian then $\mathbb{S} \cong \mathbb{Z}/p\mathbb{Z}$. Hence H is abelian p-group and we are done by case (2). $\square$

Proof. **(Theorem 4.1)** In view of Theorem 4.4, Theorem 4.1 follows from the previous theorem, once we show that there exist a solution to the embedding problem for $\pi_1^c(C)$ for the case (1), (2) and (3) of the previous theorem. The case (1) and (2) will be proved in Section 5 and (3) will be proved in Section 6. □

Before proving that a solution to embedding problem $(*)$ exists in the above three cases, we shall prove the following group theory result. This will be used in the next section.

Lemma 4.6. *Given any finite abelian p-group A there exists a finite p-group B such that the commutator of B, $[B,B]=A$.*

Proof. Since A is an abelian p-group. A is a direct sum of cyclic p-groups. Observe that the commutator of the group $B_1 \times B_2$ is isomorphic to $[B_1,B_1] \times [B_2,B_2]$ for any two groups B_1 and B_2. So we may assume A is a cyclic p-group, say $\mathbb{Z}/p^m\mathbb{Z}$. Consider the Heisenberg group over $\mathbb{Z}/p^m\mathbb{Z}$, i.e., the group of 3×3 upper triangular matrices with diagonal entries 1. It is a group of order p^{3m} generated by the matrices

$$\begin{pmatrix}1&1&0\\0&1&0\\0&0&1\end{pmatrix}, \quad \begin{pmatrix}1&0&0\\0&1&1\\0&0&1\end{pmatrix}$$

and one could easily check that the commutator of this group is the subgroup generated by

$$\begin{pmatrix}1&0&1\\0&1&0\\0&0&1\end{pmatrix}$$

which is clearly isomorphic to $(\mathbb{Z}/p^m\mathbb{Z},+)$. □

The construction of such a group using Heisenberg matrices was pointed out to me by a friend Sandeep Varma and also by Prof. Donu Arapura.

5. Quasi-p embedding problem

In this section we show that the split embedding problem has a solution if H is a perfect quasi-p group or H is a p-group. We shall begin by stating some results on quasi-p embedding problems.

Theorem 5.1. *(***Florian Pop, [Pop], [Ha3, Theorem 5.3.4], [Ha6, Corollary 4.6]***) Let k be an algebraically closed field of characteristic $p>0$, $\mathrm{card}(k)=m$, and let C be an an irreducible affine smooth curve over k. Then every quasi-p embedding problem for $\pi_1(C)$ has m distinct proper solutions.*

Theorem 5.2. *(***[Ha5, Theorem 1b]***) Let π be a profinite group such that $H^1(\pi,P)$ is infinite for every finite elementary abelian p-group P with continuous π-action. Then every p-embedding problem for π has a proper solution if and only if every p-embedding problem has a weak solution (equivalently, p cohomological dimension of π, $\mathrm{cd}_p(\pi)\le 1$).*

Theorem 5.3. *The following split embedding problem has* $\mathrm{card}(k) = m$ *proper solutions*

$$\begin{tikzcd}
& & & \pi_1^c(C) \arrow[dl, dashed] \arrow[d] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[d] & 1 \\
& & & 1 &
\end{tikzcd}$$

Here H *is a quasi-p perfect group (i.e.* $[H,H] = H$*) and* $\pi_1^c(C)$ *is the commutator of the algebraic fundamental group of an irreducible smooth affine curve* C *over an algebraically closed field* k *of characteristic* p.

Proof. As in Proposition 4.4 (also see Fig. 1), let K^{un} denote the compositum (in some fixed algebraic closure of $k(C)$) of the function fields of all étale Galois covers of C. And let K^{ab} be the subfield of K^{un} obtained by considering only abelian étale covers of C. In terms of Gaolis theory, $\pi_1^c(C)$ is $\mathrm{Gal}(K^{un}/K^{ab})$. So giving a surjection from $\pi_1^c(C)$ to G is same as giving a Galois extension $M \subset K^{un}$ of K^{ab} with Galois group G. Since K^{ab} is an algebraic extension of $k(C)$ and M is a finite extension of K^{ab}, we can find a finite abelian extension $L \subset K^{ab}$ of $k(C)$ and $L' \subset K^{un}$ a Galois extension of L with Galois group G so that $M = K^{ab}L'$. Let X be the normalization of C in L and Φ_X be the normalization morphism. Then X is an étale abelian cover of C and the function field, $k(X)$, of X is L. Let W_X be the normalization of X in L' and Ψ_X corresponding normalization morphism. Then Ψ_X is étale and $k(W_X) = L'$.

By Theorem 5.1 applied to the affine curve X and translating the conclusion into Galois theory, we conclude that there exist m distinct smooth irreducible étale Γ-covers. Each one of these Γ-cover, Z, of X is such that $Z/H = W_X$. Clearly $k(Z) \subset K^{un}$. We also have $\mathrm{Gal}(k(Z)K^{ab}/K^{ab}) \subset \Gamma$ and by assumption $\mathrm{Gal}(k(W_X)K^{ab}/K^{ab}) = G$. Moreiver, Galois group of $k(Z)/k(W_X)$ is H which is a perfect group and $k(W_X)K^{ab}/k(W_X)$ is a pro-abelian extension. Hence they are linearly disjoint, so $\mathrm{Gal}(k(Z)K^{ab}/k(W_X)K^{ab}) = H$. So $\mathrm{Gal}(k(Z)K^{ab}/K^{ab}) = \Gamma$. Also if Z and Z' are two distinct solutions then the $\mathrm{Gal}(k(Z)k(Z')/k(W_X))$ is quotient of $H \times H$ and hence perfect. So $\mathrm{Gal}(k(Z)k(Z')K^{ab}/k(W_X)K^{ab} = \mathrm{Gal}(k(Z)k(Z')/k(W_X))$ and consequently $k(Z)K^{ab}$ and $k(Z')K^{ab}$ are distinct fields. □

Now we shall consider the case when H is an abelian p-group.

Lemma 5.4. *Let* P *be any nonzero finite abelian p-group, then there exist* m *distict surjections from* $\pi_1^c(C)$ *to* P.

Proof. Let $n \geq 1$ be a natural number. By Lemma 4.6 there exist a p-group P_1 such that its commutator $[P_1, P_1] = P$. By Theorem 5.1 there exist m distinct surjective homomorphisms from $\pi_1(C)$ to P_1 and clearly the commutator $\pi_1^c(C)$ surjects onto $[P_1, P_1] = P$ under their restrictions.

Theorem 5.5. *The following split embedding problem has* $\mathrm{card}(k) = m$ *proper solutions*

$$\begin{tikzcd} & & & \pi_1^c(C) \arrow[dl, dashed] \arrow[d] & \\ 1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[d] & 1 \\ & & & 1 & \end{tikzcd}$$

Here H is a minimal normal subgroup of Γ and an abelian p-group and $\pi_1^c(C)$ is the commutator of the algebraic fundamental group of a smooth affine curve C over an algebraically closed field k of characteristic p.

Proof. Since $Z(\Gamma)$ the center of Γ and H are both normal subgroup of Γ, so is $Z(\Gamma) \cap H$. Since H is minimal normal subgroup of Γ, $Z(\Gamma) \cap H$ is trivial or $H \subset Z(\Gamma)$. If $H \subset Z(\Gamma)$ then Γ acts trivially on H, hence $\Gamma \cong G \times H$. By previous lemma there are m surjections of $\pi_1^c(C)$ onto H. Borrowing the notation from Theorem 5.3, we conclude that there are m different field extensions of K^{ab} contained in K^{un} with Galois group H. M being a finite field extension of K^{ab} only finitely many of these H-extensions are not linearly disjoint with M over K^{ab}. Hence there are m H-extensions of K^{ab} which are linearly disjoint with M over K^{ab} and compositum of each of these H-extensions with M lead to a Γ-extension of K^{ab}. Hence we have m solution to the embedding problem. Now suppose $Z(\Gamma) \cap H$ is trivial, i.e. Γ acts on H nontrivially. By Theorem 5.1 there are m proper solutions to the embedding problem for $\pi_1(X)$ where X is as in the proof of Theorem 5.3. So there are m distinct smooth irreducible étale H-cover of W_X which are Γ-covers of X. For each such H-cover Z, we shall show that $\mathrm{Gal}(k(Z)K^{ab}/K^{ab})$ is isomorphic to Γ. Suppose not, then $k(Z)$ is not linearly disjoint with $M = k(W_X)K^{ab}$ over $k(W_X)$. So there exists a nontrivial field extension $L''/k(W_X)$ with $L'' = k(Z) \cap k(W_X)K^{ab}$. So $L'' = Kk(W_X)$ for K some finite field extension of $k(X)$ and $K \subset K^{ab}$. $K^{ab}/k(X)$ is a pro-abelian extension, so $K/k(X)$ is a Galois extension (with in fact abelian Galois group). Hence $L''/k(X)$ is a Galois extension. So we conclude that $\mathrm{Gal}(k(Z)/L'')$ is a normal subgroup of $\Gamma = \mathrm{Gal}(k(Z)/k(X))$, but $\mathrm{Gal}(k(Z)/L'') \subset H = \mathrm{Gal}(k(Z)/k(W_X))$. H being minimal normal subgroup of Γ and $L''/k(W_X)$ being nontrivial extension, forces $L'' = k(Z)$ and hence $\mathrm{Gal}(K/k(X)) = H$. But this contradicts the fact that Γ acts on H nontrivially. Now if Z and Z' are two distinct H-covers, replace the field $k(Z)$ by $k(Z)k(Z')$ and Γ by $\mathrm{Gal}(k(Z)k(Z')/k(X))$ in the above argument to conclude that $k(Z)k(Z')$ is linearly disjoint with M over $k(W_X)$. Hence $k(Z)K^{ab}$ and $k(Z')K^{ab}$ are distinct field extensions of M. □

Below we give an alternative approach to asserting existence of atleast one proper solution of the embedding problem for $\pi_1^c(C)$ when H is any p-group. This is a cohomological approach.

Theorem 5.6. *The following split embedding problem has a proper solution*

$$\begin{tikzcd}
 & & & \pi_1^c(C) \arrow[dl, dashed] \arrow[d] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd}$$

Here H is a p-group and $\pi_1^c(C)$ is the commutator of the algebraic fundamental group of a smooth affine curve C over an algebraically closed field k of characteristic p.

Proof. This theorem will follow trivially from Theorem 5.2, once we prove that $H^1(\pi_1^c(C), P)$ is infinite for every elementary abelian p-group P with continuous $\pi_1^c(C)$-action, since we have already seen that $\pi_1^c(C)$ is a projective profinite group, hence its cohomological dimension is less than or equal to 1. $H^1(\pi_1^c(C), P)$ is infinite is shown in Proposition 5.7 below. □

Proposition 5.7. *Let P be any nonzero finite elementary abelian p-group with a continuous action of $\pi_1^c(C)$, then the first group cohomology $H^1(\pi_1^c(C), P)$ is infinite.*

Proof. Let Φ be the kernel of the action of $\pi_1^c(C)$ on P. Then Φ is a normal subgroup of $\pi_1^c(C)$ of finite index. We know $\pi_1^c(C)$ acts on the K^{un} and has fixed field K^{ab}. Let M be the fixed field of Φ, then $\mathrm{Gal}(M/K^{ab}) = \pi_1^c(C)/\Phi$. Since M is a finite extension of K^{ab}, there exists L a finite abelian extension of $k(C)$ and L' a finite extension of L such that $\mathrm{Gal}(L'/L) = \mathrm{Gal}(M/K^{ab})$ and $L'K^{ab} = M$. Let X be the normalization of C in L and Y be the normalization of C in L'. If we translate this Galois theory to Galois groups, we get the following commutative diagram:

$$\begin{tikzcd}
\Phi \arrow[r, hook] \arrow[d, hook] & \pi_(Y) \arrow[d, hook] \\
\pi_1^c(C) \arrow[r, hook] \arrow[d] & \pi_1(X) \arrow[d] \\
\pi_1^c(C)/\Phi \arrow[r, "\sim"] & \pi_1(X)/\pi_1(Y)
\end{tikzcd}$$

So we notice that $\pi_1(X) = \pi_1^c(C)\pi_1(Y)$. Now we define an action of $\pi_1(X)$ on P by defining it to be trivial on $\pi^1(Y)$ and to be the given action on $\pi_1^c(C)$. This is well defined because $\pi_1^c(C) \cap \pi_1(Y) = \Phi$, which is the kernel of the action of $\pi_1^c(C)$ on P. Now consider the following short exact sequence of groups:

$$1 \to \pi_1^c(C) \to \pi_1(X) \to \Pi \to 1$$

where Π is simply the quotient $\pi_1(X)/\pi_1^c(C)$. Applying Hoschild-Serre spectral sequence for group cohomology [Wei 7.5.2] to this short exact sequence, we get the following long exact sequence:

$$0 \to H^1(\Pi, H^0(\pi_1^c(C), P)) \to H^1(\pi_1^c(X), P) \to H^0(\Pi, H^1(\pi_1^c(C), P))$$

$$\to H^2(\Pi, H^0(\pi_1^c(C), P))$$

If the action of $\pi_1^c(C)$ on P is such that it fixes only 0, then $H^0(\pi_1^c(C), P) = 0$, hence the first and the fourth term in the above long exact sequence is 0. Also we know that the $H^1(\pi_1^c(X), P)$ is infinite by [Ha4, Proposition 3.8]. So the $H^1(\pi_1^c(C), P) \supset H^0(\Pi, H^1(\pi_1^c(C), P))$ is infinite. So we may assume $P^{\pi_1^c(C)}$ is nonzero. In this case we have a short exact sequence of $\pi_1^c(C)$-modules:

$$0 \to P^{\pi_1^c(C)} \to P \to P/P^{\pi_1^c(C)} \to 0$$

Here $\pi_1^c(C)$ acts trivially on the first term and fixes nothing in the third term, i.e., $H^0(\pi_1^c(C), P/P^{\pi_1^c(C)}) = 0$, so we get the long exact sequence of group cohomology which looks like:

$$... \to H^0(\pi_1^c(C), P/P^{\pi_1^c(C)}) = 0 \to H^1(\pi_1^c(C), P^{\pi_1^c(C)}) \to H^1(\pi_1^c(C), P)...$$

Since $\pi_1^c(C)$ acts trivially on $P^{\pi_1^c(C)}$, $H^1(\pi_1^c(C), P^{\pi_1^c(C)}) = \mathrm{Hom}(\pi_1^c(C), P^{\pi_1^c(C)})$. And we know that $\mathrm{Hom}(\pi_1^c(C), P^{\pi_1^c(C)})$ is infinite by previous lemma. So we conclude that $H^1(\pi_1^c(C), P)$ is infinite. □

Remark: This alternative method could be possibly made strong enough to yield m solutions if one could prove a refined version of Theorem 5.2 relating the cardinality of $H^1(\pi, P)$ with the cardinality of distinct proper solutions for p-group embedding problem for π. This looks plausible since the two method seems similar in spirit.

6. Prime-to-p embedding problem

In this section, we prove certain results on formal patching and then use them to solve the prime-to-p embedding problems for $(*)$. We begin with some patching results (6.1, 6.2, 6.3) which roughly mean the following: suppose we have a proper $k[[t]]$-scheme T whose special fiber is a collection of smooth irreducible curves intersecting at finitely many points. Finding a cover of T is equivalent to finding a cover of these irreducible curves away from those finitely many intersection points and covers of formal neighbourhood of the intersection points so that they agree in the punctured formal neighbourhoods of the intersection points. In our situation, the special fiber of T is connected sum of X and N copies of Y. Each copy of Y intersects X at a point r_i of X and a point s of Y for $1 \leq i \leq n$. Now suppose we have an irreducible G-cover, $\Psi_X : W_X \to X$ étale at $r_1, .., r_n$ and an irreducible H-cover, $\Psi_Y : W_Y \to Y$ étale at s then we construct a Γ covering of T by patching a Γ-cover of X, $\mathrm{Ind}_G^\Gamma W_X = (\Gamma \times W_X)/\sim$, where $(\gamma, w) \sim (\gamma g^{-1}, gw)$ for $\gamma \in \Gamma$, $g \in G$ and w a point of W_X, and a Γ-cover of Y, $\mathrm{Ind}_H^\Gamma W_Y$. This is possible since both these covers restrict to Γ-covers induced from trivial cover in the formal punctured neighbourhood of the intersection points, so we can pick trivial Γ-covers of the intersection points which obviously will restrict to trivial Γ-cover on the punctured neighbourhood. Now we proceed to show how all this works. We start with the following patching result.

Theorem 6.1. *(***[Ha3, Theorem 3.2.12]***) Let $(A, \mathfrak{p})$ be a complete local ring and let T be a proper A-scheme. Let $\{\tau_1, .., \tau_N\}$ be a set of closed points of T and $T^o = T \setminus \{\tau_1, .., \tau_N\}$. Let $\hat{T}_i = \mathrm{Spec}(\hat{\mathcal{O}}_{T,\tau_i})$, $\widetilde{T^o}$ be the $\mathfrak{p}$-adic completion of T^o and $\mathcal{K}_i$ be the $\mathfrak{p}$-adic completion of $\hat{T}_i \setminus \{\tau_i\}$. Then the base change functor*

$$\mathcal{M}(T) \to \mathcal{M}(\widetilde{T^o}) \times_{\mathcal{M}(\cup_{i=1}^N \mathcal{K}_i)} \mathcal{M}(\cup_{i=1}^N \hat{T}_i)$$

is an equivalence of categories. And this remains true with $\mathcal{M}$ replaced by $\mathcal{AM}$, $\mathcal{SM}$ or $G\mathcal{M}$ for any finite group G.

In fact [Ha3, Theorem 3.2.12] is even stronger and allows one to assert the equivalence of categories even if one replaces T, T^o, etc. with their pull back by a proper morphism. The proof uses Grothendieck's Existence Theorem and a result of Ferrand-Raynaud or rather its genralization by M. Artin ([Ha3, Theorem 3.1.9]). The latter asserts, for a noetherian scheme T, the equivalence of categories between $\mathcal{M}(T)$ and $\mathcal{M}(T^o) \times_{\mathcal{M}(W^o)} \mathcal{M}(\hat{W})$ where W is a finite set of closed points of T, $T^o = T \setminus W$, $\hat{W}$ is the completion of T along W and $W^o = \hat{W} \times_T T^o$.

Now we shall specialize to what we need. Let k be a field. Let X and Y be irreducible smooth projective k-curves with finite k-morphisms $\Phi_X : X \to \mathbb{P}^1_x$ and $\Phi_Y : Y \to \mathbb{P}^1_y$, where $\mathbb{P}^1_x$ and $\mathbb{P}^1_y$ are projective lines with local coordinate x and y respectively. Also assume that Φ_Y is totally ramified at $y = 0$. Let R and S be such that $\operatorname{Spec}(R) = X \setminus \Phi_X^{-1}(\{x = \infty\})$ and $\operatorname{Spec}(S) = Y \setminus \Phi_Y^{-1}(\{y = \infty\})$. So $k[x] \subset R$ and $k[y] \subset S$. Let $A = (R \otimes_k S \otimes_k k[[t]])/(t - xy)$ and $T^a = \operatorname{Spec}(A)$. Let T be the closure of T^a in $X \times_k Y \times_k \operatorname{Spec}(k[[t]])$. Let L be an affine line $\operatorname{Spec}(k[z])$. The k-algebra homomorphism $k[[t]][z] \to A$ given by $z \mapsto x + y$ induces a $k[[t]]$-morphism ϕ from T^a to $L^* = L \times_k k[[t]]$. Let $\lambda \in L$ be the closed point $z = 0$. L is contained in L^* as the special fiber, so λ when viewed as a closed point of L^* corresponds to the maximal ideal (z, t) in $k[[t][z]$. Let $\phi^{-1}(\lambda) = \{\tau_1, .., \tau_N\} \subset T^a$. Note that the special fiber of T is a reducible curve consisting of X and N copies of Y, each copy of Y intersecting X at $\tau_1, .., \tau_N$ since locus of $t = 0$ is same as $xy = 0$ in T and the locus of $t = 0$ and $x + y = 0$ is same as the locus of $x = 0$ and $y = 0$. Let r_i denote the point of X corresponding to τ_i, so $\Phi_X^{-1}(x = 0) = \{r_1, .., r_N\}$ and s denote the point on each copy of Y corresponding to τ_i, so s is the unique point of Y lying above $y = 0$. Borrowing notation from the previous lemma, let $T^o = T \setminus \{\tau_1, ..\tau_N\}$ and $X^o = X \setminus \{r_1, .., r_N\}$. Let $\hat{T}_i = \operatorname{Spec}(\hat{\mathcal{O}}_{T,\tau_i})$ and Let $T_X = T^o \setminus \{x = 0\}$ which is the same as the closure of $\operatorname{Spec}(A[1/x])$ in $X^o \times_k Y \times_k \operatorname{Spec}(k[[t]])$. Similarly, define $T_Y = T^o \setminus \{y = 0\}$.

Recall that $\hat{K}_{X,r_i}$ is the quotient field of $\hat{\mathcal{O}}_{X,r_i}$. Define $\mathcal{K}_{X,r_i} = \operatorname{Spec}(\hat{K}_{X,r_i}[[t]] \otimes_{k[y]} \mathcal{O}_{Y,s})$ where we regard $\hat{K}_{X,r_i}[[t]]$ as $k[y]$-module via the homomorphism which sends y to t/x. Similarly, define $\mathcal{K}_Y^i = \operatorname{Spec}(\hat{K}_{Y,s}[[t]] \otimes_{\operatorname{Spec}(k[x])} \mathcal{O}_{X,r_i})$, where we regard $\hat{K}_{Y,s}[[t]]$ as $k[x]$-module via the homomorphism which sends x to t/y. Let x_i be a local coordinate of X at r_i and y_0 be a local coordinate of Y at s.

With these notations we shall deduce the following result from Theorem 6.1. This result is analogous to [Ha2, Corollary 2.2].

Lemma 6.2. *The base change functor*

$$\mathcal{M}(T) \to \mathcal{M}(\widetilde{T_X} \cup \widetilde{T_Y}) \times_{\mathcal{M}(\cup_{i=1}^N (\mathcal{K}_{X,r_i} \cup \mathcal{K}_Y^i))} \mathcal{M}(\cup_{i=1}^N \hat{T}_i)$$

is an equivalence of categories. Moreover, same assertion holds if one replaces $\mathcal{M}$ by $\mathcal{AM}$, $\mathcal{SM}$ and $G\mathcal{M}$ for a finite group G.

Proof. First of all we observe that the closed fiber of T^o, which is the subscheme defined by the ideal (t), is disconnected. Since closed fiber of $T_X \cup T_Y$ is the closed fiber of T^o and the closed fibers of T_X and T_Y, as a subset of the closed fiber of T^o, are open and disjoint. So if we consider their (t)-adic completion we get $\widetilde{T^o} = \widetilde{T_X} \cup \widetilde{T_Y}$. Similarly the punctured spectrum $\hat{T}_i \setminus \{\tau_i\}$ is the spectrum of

the ring $k[[x_i, y_0]][(x+y)^{-1}]$). Since the only prime ideals of $k[[x_i, y_0]][(x+y)^{-1}]$ containing (t) are (x_i) and (y_0), we may first localize $k[[x_i, y_0]][(x+y)^{-1}]$ with respect to the complement of $(x_i) \cup (y_0)$ then take the (t)-adic completion. Now using [Mat, 8.15], we get that the (t)-adic completion of $\hat{T}_i \setminus \{\tau_i\}$ is $\mathcal{K}_{X,r_i} \cup \mathcal{K}^i_Y$. $\square$

Let G and H be subgroups of a finite group Γ, such that $\Gamma = G \rtimes H$.

Proposition 6.3. *Under the notation and assumption of previous lemmas, let $\Psi_X : W_X \to X$ be an irreducible normal G-cover étale over $r_1, .., r_N$. and $\Psi_Y : W_Y \to Y$ be an irreducible normal H-cover étale over s. Let W_{XT} be the normalization of an irreducible dominating component of $W_X \times_X T$ and similarly W_{YT} be the normalization of an irreducible dominating component of $W_Y \times_Y T$. Then there exists an irreducible normal Γ-cover $W \to T$ such that*
(1) $W \times_T \widetilde{T_X} = \mathrm{Ind}_G^\Gamma \widetilde{W_{XT} \times_T T_X}$
(1') $W \times_T \widetilde{T_Y} = \mathrm{Ind}_H^\Gamma \widetilde{W_{YT} \times_T T_Y}$
(2) $W \times_T \hat{T}_i$ is a Γ-cover of $\hat{T}_i$ induced from the trivial cover.
(3) $W \times_T \mathcal{K}_{X,r_i}$ is a Γ-cover of $\mathcal{K}_{X,r_i}$ induced from the trivial cover.
(4) $W \times_T \mathcal{K}^i_Y$ is a Γ-cover of $\mathcal{K}^i_Y$ induced from the trivial cover.
(5) $W/H \cong W_{XT}$ as a cover of T.

Proof. Let $\widetilde{W_X} = \mathrm{Ind}_G^\Gamma \widetilde{W_{XT} \times_T T_X}$ and $\widetilde{W_Y} = \mathrm{Ind}_H^\Gamma \widetilde{W_{YT} \times_T T_Y}$. So $\widetilde{W_X}$ and $\widetilde{W_Y}$ are Γ-covers of $\widetilde{T_X}$ and $\widetilde{T_Y}$ respectively. Hence their union, $\widetilde{W^o}$, is an object of $\Gamma\mathcal{M}(\widetilde{T_X} \cup \widetilde{T_Y})$. Now for each i, $\widetilde{W_X} \times_{\widetilde{T_X}} \mathcal{K}_{X,r_i} = \mathrm{Ind}_G^\Gamma W_X \times_X \mathcal{K}_{X,r_i}$. But $W_X \times_X \mathcal{K}_{X,r_i}$ is $\mathrm{card}(G)$ copies of $\mathcal{K}_{X,r_i}$, since W_X is étale over r_i. And similarly, $\widetilde{W_Y} \times_{\widetilde{T_Y}} \mathcal{K}^i_Y = \mathrm{Ind}_H^\Gamma W_Y \times_Y \mathcal{K}^i_Y$ which is Γ copies of $\mathcal{K}^i_Y$, since $W_Y \to Y$ is étale over s. Hence $\widetilde{W^o}$ restricted to $\cup_{i=1}^N (\mathcal{K}_{X,r_i} \cup \mathcal{K}^i_Y)$ is a Γ-cover induced from the trivial cover. Let $\hat{W}_i$ be a Γ-cover of $\hat{T}_i$ induced from the trivial cover. Then their union, $\hat{W}$, is an object in $\Gamma\mathcal{M}(\cup_{i=1}^N \hat{T}_i)$ which when restricted to $\cup_{i=1}^N (\mathcal{K}_{X,r_i} \cup \mathcal{K}^i_Y)$ obviously is a Γ-cover induced from the trivial cover. So after fixing an isomorphism between the two trivial Γ-covers of $\cup_{i=1}^N (\mathcal{K}_{X,r_i} \cup \mathcal{K}^i_Y)$, we can apply the above patching lemma and obtain an object W in $\Gamma\mathcal{M}(T)$ which induces the covers $\widetilde{W^o}$ and $\hat{W}$ on $\widetilde{T^o}$ and $\cup_{i=1}^N \hat{T}_i$ respectively. Hence we get conclusion (1) to (4) of the proposition. So it remains to prove W is irreducible and normal and conclusion (5) holds. For irreducibility of W we note that G and H generate Γ. Suppose W is reducible. Consider Γ^o, the stabilizer of the identity component of W. So W has $\mathrm{card}(\Gamma/\Gamma^o)$ irreducible component. Since G is the stabilizer of the identity component of $\widetilde{W_X}$, and H is the stabilizer of the identity component of $\widetilde{W_Y}$, G and H is contained in Γ^o. Hence $\Gamma^o = \Gamma$. Hence W is irreducible. To show W is normal it is enough to show that for each closed point σ of T, $W_\sigma = W \times_T \mathrm{Spec}(\hat{\mathcal{O}}_{T,\sigma})$ is normal. If $\sigma = \tau_i$ for some i then W_σ is isomorphic to copies of $\hat{T}_i$ and hence is normal. Otherwise σ belongs to T_X (or T_Y). So W_σ is isomorphic to $\mathrm{Ind}_G^\Gamma \widetilde{W_{XT} \times_T T_X} \times_{T_X} \mathrm{Spec}(\hat{\mathcal{O}}_{T_X,\sigma})$, which is a union of copies of $\mathrm{Spec}(\hat{\mathcal{O}}_{W_{XT} \times_T T_X, \sigma'})$, where σ' are points of $W_{XT} \times_T T_X$ lying above σ. But $W_{XT} \times_T T_X$ is normal. Similar argument holds in the case when $\sigma \in T_Y$. Next we shall show that W/H and W_{XT} restricts to same G-cover on the patches $\widetilde{T_X}$, $\widetilde{T_Y}$ and $\hat{T}_i$ for all i. So conclusion (5) will follow from the previous lemma's assertion about the equivalence of categories. Clearly, both W/H and W_{XT} restricts to trivial G-cover of $\hat{T}_i$. Now, $W_{XT} \times_T \widetilde{T_X} = W_{XT} \times_T \widetilde{(T)} \times_{\widetilde{T}} \widetilde{T_X} = \widetilde{W_{XT}} \times_{\widetilde{T}} \widetilde{T_X}$ and this is same as $\widetilde{W_{XT} \times_T T_X}$ since T_X is an open subscheme of

T. On the other hand $W/H \times_T \widetilde{T_X} = (W \times_T \widetilde{T_X})/H$. But by (1), this is same as $\widetilde{W_{XT} \times_T} T_X$. Finally, $W_{XT} \times_T \widetilde{T_Y} = \mathrm{Ind}_{\{e\}}^G \widetilde{T_Y}$, since the image of T_Y under the map $T \to X$, is the generic point. So the G-cover $W_{XT} \to T$ is trivial over the subscheme T_Y. And by (1′), $W/H \times_T \widetilde{T_Y} = (\mathrm{Ind}_H^\Gamma \widetilde{W_{TY}})/H$ which is same as $\mathrm{Ind}_{H/H}^{\Gamma/H} \widetilde{T_Y}$ since $W_Y/H = Y$. But Γ/H is G. □

Lemma 6.4. *Let T, X and Y be as in previous lemma. Let D be an irreducible smooth projective k-curve. Assume that $\Phi_X : X \to \mathbb{P}^1_x$ factors through D, i.e., there exist $\Phi'_X : X \to D$ and $\Theta : D \to \mathbb{P}^1_x$ such that their composition is Φ_X. Also assume Φ_Y and Φ'_X are abelian covers. For any $k[[t]]$-scheme V, let V^g denote the generic fiber. Then the morphism $\Phi'_X \times \Phi_Y \times Id_{\mathrm{Spec}(k[[t]])}$ restricted to T from T to its image in $D \times_k \mathbb{P}^1_y \times_k \mathrm{Spec}(k[[t]])$ induces an abelian cover of projective $k((t))$-curves $T^g \to D \times_k \mathrm{Spec}(k((t)))$.*

Proof. We need to show that the function field, $k(T)$, of T is an abelian extension of the field $k(D) \otimes_k k((t))$. Note that $k(T)$ is the compositum of $L_1 = k(X) \otimes_k k((t))$ and L_2. Here L_2 is the function field of a dominating irreducible component of

$$(Y \times_k \mathrm{Spec}(k((t)))) \times_{\mathbb{P}^1_y \times_k \mathrm{Spec}(k((t)))} (D \times_k \mathrm{Spec}(k((t))))$$

where the morphism $D \times_k \mathrm{Spec}(k((t))) \to \mathbb{P}^1_y \times_k \mathrm{Spec}(k((t)))$ is the composition of $D \times_k \mathrm{Spec}(k((t))) \to \mathbb{P}^1_x \times_k \mathrm{Spec}(k((t)))$ with morphism $\mathbb{P}^1_x \times_k \mathrm{Spec}(k((t))) \to \mathbb{P}^1_y \times_k \mathrm{Spec}(k((t)))$ defined in local coordinates by sending y to t/x. Since L_1 is a base change of finite extension of $k(x)$ by $k(D) \otimes_k k((t))$ and L_2 is a base change of finite extension of the subfield $k(t/x)$ by $k(D) \otimes_k k((t))$, we have $L_1 \cap L_2 = k(D) \otimes_k k((t))$. Hence L_1 and L_2 are linearly dijoint over $k(D) \otimes_k k((t))$. Now $\mathrm{Gal}(L_1/k(D) \otimes_k k((t)))$ is isomorphic to $\mathrm{Gal}(k(X)/k(D))$ and $\mathrm{Gal}(L_2/k(D) \otimes_k k((t)))$ is isomorphic to $\mathrm{Gal}(k(Y)/k(y))$. Hence these groups are abelian, since the latter groups are so. Using the fact that the Galois group of compositum of linearly disjoint Galois field extensions is the direct sum of the two Galois groups, we get that $\mathrm{Gal}(k(T)/k(D) \otimes_k k((t)))$ is a direct sum of abelian groups, hence is abelian. □

We shall see a variation of the following result, which is a special case of [Ha2, Proposition 2.6, Corollary 2.7].

Proposition 6.5. *(**Harbater**) Let k be an algebraically closed field. Let X_0^s be a smooth projective connected smooth k-curve. Let $\zeta^1, .., \zeta^r \in X_0^s$. Let X_0 and X_1 be irreducible normal projective $k[[t]]$-curves. Suppose X_1 has geometrically smooth closed fiber. Let $\psi : X_1 \to X_0$ be a G-cover with generic fiber $\psi^g : X_1^g \to X_0^g$. Assume $X_0 = X_0^s \times_k \mathrm{Spec}(k[[t]])$ and X_1^g has genus at least 1. Also assume ψ^g is a smooth G-cover étale away from $\{\zeta_1, .., \zeta_r\}$ where $\zeta_j = \zeta^j \times_k k((t)) \in X_0^g$ for $1 \le j \le r$. Then there exist smooth connected G-cover $\psi^s : X_1^s \to X_0^s$ étale away from $\{\zeta^1, .., \zeta^r\}$.*

The proof of the following result is also similar to the one given in [Ha2]. Though in [Ha2] the assumption that X_1 is a nonconstant family and hence the assertion of existence of m distinct solutions is not made, it is possible to do this as we shall see below.

Proposition 6.6. *Let k be an algebraically closed field. Let $X_0, .., X_3$ be irreducible normal projective $k[[t]]$-curves and for $i > 0$, X_i have generically smooth closed fibers. For $i = 1, 2$ and 3, let $\psi_i : X_i \to X_{i-1}$ be proper surjective $k[[t]]$-morphisms*

and $\psi_i^g : X_i^g \to X_{i-1}^g$ be the induced morphisms on the generic fibers. Assume $X_0^g = X_0^s \times_k k((t))$ for some smooth projective k-curves X_0^s and X_1^g is of genus atleast 1. Let $\zeta^1, .., \zeta^r \in X_0^s$ and $\zeta_j = \zeta^j \times_k k((t)) \in X_0^g$ for $1 \le j \le r$, so that $\psi_1^g \circ \psi_2^g \circ \psi_3^g$ is étale away from $\{\zeta_1, .., \zeta_r\}$. Let ψ_1 be an A-cover, ψ_2 be a G-cover, ψ_3 be an H-cover and $\psi_2 \circ \psi_3$ be a Γ-cover. Then there exist X_1^s, X_2^s and X_3^s connected smooth projective k-curves and morphisms $\psi_i^s : X_i^s \to X_{i-1}^s$ so that $\psi_1^s \circ \psi_2^s \circ \psi_3^s$ is étale away from $\{\zeta^1, .., \zeta^r\}$ and ψ^1 is an A-cover, ψ^2 is a G-cover, ψ^3 is an H-cover and $\psi^2 \circ \psi^3$ is a Γ-cover.

Proof. Since all the three groups are finite, the covers ψ_i for $i = 1, .., 3$ descends to a B-morphism, where $B \subset k[[t]]$ is a regular finite type $k[t]$-algebra. That is, there exist connected B-schemes X_i^B and morphism $\psi_i^B : X_i^B \to X_{i-1}^B$ where ψ_1^B is an A-cover, ψ_2^B is a G-cover, ψ_3^B is an H-cover and $\psi_2^B \circ \psi_3^B$ is a Γ-cover and ψ_i^B induces ψ_i. Moreover for $E = \mathrm{Spec}(B[t^{-1}])$, $X_i^E = X_i^B \times_B E$ are regular and X_0^E is isomorphic to $X_0^s \times_k E$. The induced morphism ψ_i^E are such that ψ_1^E is an A-cover, ψ_2^E is a G-cover, ψ_3^E is an H-cover, $\psi_2^E \circ \psi_3^E$ is a Γ-cover and $\psi_1^E \circ \psi_2^E \circ \psi_3^E$ is ramified only over $\{\zeta_E^1, ..\zeta_E^r\}$. To complete the proof, we shall show that there exists a nonempty open subset E' of E so that the fiber of ψ_i^E over each closed point of E' is irreducible and nonempty. First we note that by [Ha2, Lemma 2.4(b)] the closed fibers of $X_i \to \mathrm{Spec}(k[[t]])$ are connected, since by assumption the closed fibers are generically smooth. Hence the fibers of ψ_i^B over $(t = 0)$ are connected because X_i^B induces X_i. Since X_i's are normal, X_i^B's are unibranched along the corresponding fibers over $(t = 0)$. Hence by [Ha1, Proposition 5], we have a nonempty open subset of $\mathrm{Spec}(B)$, and hence subset E' of $E = \mathrm{Spec}(B) \setminus (t = 0)$, such that for all closed points $e \in E'$ the fibers X_i^e of $X_i^E \to E$ over e are irreducible. Next, we shall show that there exist a nonempty open subset S of E' such that the restriction morphism $X_i^S \to S$ is smooth of relative dimension 1. Since k is algebraically closed $k(X_0^s)$ is seperably generated over k. Hence $k(X_0^{E'})$ is seperably generated over $k(E')$. Moreover, since ψ_i^E are finite seperable morphisms (in fact their composition is étale away from $\{\zeta_E^1, ..\zeta_E^r\}$), we have $k(X_0^{E'})$ is seperably generated over $k(E')$. Since $X_i^{E'} \to E'$ is a morphism integral schemes of relative dimesion 1 and is generically seperable, the relative sheaf of differentials is free of rank 1 at the generic point ([Eis, Corollary 16.17a]). Hence there exist an open subset S of E' such that the morphism $X_i^S \to S$ is smooth of relative dimension 1. Moreover, the fiber X_i^s at each point $s \in S \subset E'$ is irreducible. □

Lemma 6.7. *There exists an abelian cover $Y \to \mathbb{P}_y^1$ ramified only at $y = 0$, where it is totally ramified, with genus of Y arbitrarily large.*

Proof. Let Y' be the normal cover of $\mathbb{P}_y^1$ defined by the equation $u^{p^n} - u - y^{p^n+1} = 0$. To see this is an irreducible polynomial in $k(y)[u]$, by Gauss lemma, it is enough to show it is irreducible in $k[y, u]$. But in fact, it is irreducible in $k(u)[y]$ since $p^n + 1^{th}$ root of $u^{p^n} - u$ does not belong to $k(u)$. Also Y' is étale everywhere except $y = \infty$ and since there is only one point in Y' lying above ∞ it is totally ramified there. So by translation we can get Y, a cover of $\mathbb{P}_y^1$, which is totally ramified at $y = 0$ and étale elsewhere. Also the genus of Y, by the Hurwitz formula, is given by the equation $2g(Y) - 2 = (p^n + 1)(g(\mathbb{P}_u^1) - 2) + \deg(R)$ where R is the ramification divisor of the morphism $Y \to \mathbb{P}_u^1$. We also know that $\deg(R) = \sum_{P \in Y} e_P - 1$ where e_P is the ramification index at the point $P \in Y$. Now branch locus of Y as a cover of $\mathbb{P}_u^1$ is given by $u^{p^n} - u = 0$ and $u = \infty$. For each point $P \in Y$ lying above a

branch point other than ∞, $e_P = p^n + 1$, so we get that $\deg(R) \geq p^n p^n$. So we get the following inequality for genus of Y.

$$2g(Y) - 2 \geq -2(p^n + 1) + p^{2n}$$
$$\Rightarrow g(Y) \geq p^n(p^n - 2)/2$$

Clearly $g(Y)$ could be made arbitrary large. Also note that $\mathrm{Gal}(k(Y)/k(y)) \cong \bigoplus_{i=1}^{n} \mathbb{Z}/p\mathbb{Z}$. Hence Y is an abelian cover of $\mathbb{P}^1_y$. $\square$

Theorem 6.8. *The following split embedding problem has a proper solution*

$$\begin{tikzcd}
 & & & \pi_1^c(C) \arrow[dl, dashed] \arrow[d] & \\
1 \arrow[r] & H \arrow[r] & \Gamma \arrow[r] & G \arrow[r] \arrow[d] & 1 \\
 & & & 1 &
\end{tikzcd}$$

Here H is prime to p-group minimal normal subgroup of Γ and $\pi_1^c(C)$ is the commutator of the algebraic fundamental group of a smooth affine curve C over an algebraically closed field k of characteristic p.

Proof. Let K^{un} denote the compositum (in some fixed algebraic closure of $k(C)$) of the function field of all Galois étale covers of C. And let K^{ab} be the subfield of K^{un} obtained by considering only abelian covers with above property. In these terms $\pi_1^c(C)$ is $\mathrm{Gal}(K^{un}/K^{ab})$. So giving a surjection from $\pi_1^c(C)$ to G is same as giving a Galois extension $M \subset K^{un}$ of K^{ab} with Galois group G. Since K^{ab} is an algebraic extension of $k(C)$ and M is a finite extension of K^{ab}, we can find a finite abelian extension $L \subset K^{ab}$ of $k(C)$ and $L' \subset K^{un}$ a G-Galois extension of L so that $M = K^{ab}L'$. Let D be the smooth compactification of C, X be the normalization of D in L and $\Phi'_X : X \to D$ be the normalization morphism. Then X is an abelian cover of D étale over C and the function field, $k(X)$, of X is L. Let W_X be the normalization of X in L' and Ψ_X corresponding normalization morphism. Then Ψ_X is étale away from points lying above $D \setminus C$ and $k(W_X) = L'$. Since k is algebraically closed, $k(C)/k$ has a seperating transedence basis. By a stronger version of Noether normalization (for instance, see [Eis, Corollary 16.18]), there exist a finite proper k-morphism from C to $\mathbb{A}^1_x$, where x denotes the local coordinate of the affine line, which is generically seperable. The branch locus of such a morphism is codimension 1, hence this morphism is étale away from finitely many points. By translation we may assume none of these points map to $x = 0$. This morphism extends to a finite proper morphism $\Theta : D \to \mathbb{P}^1_x$. Let $\Phi_X : X \to \mathbb{P}^1_x$ be the composition $\Theta \circ \Phi'_X$. Let $\{r_1, .., r_N\} = \Phi_X^{-1}(\{x = 0\})$, then Φ_X is étale at $r_1, .., r_N$. Also note that $\Theta^{-1}(\{x = \infty\}) = D \setminus C$. Let $l > 0$ be any integer. Let $\Phi_Y : Y \to \mathbb{P}^1_y$ be an abelian cover étale everywhere except $y = 0$, where it is totally ramified and genus of Y is at least 2 and more than the number of generators for H^l, i.e., product of H with itself l times. Let s be the point lying above $y = 0$. Existence of such a Y is guaranteed by Lemma 6.7. Since H^l is prime-to-p, and Y has high genus by [SGAI, XIII, Corollary 2.12, page 392], there exists irreducible étale H^l-cover W'_Y of Y. By taking appropriate quotient we get l distinct étale H-covers of Y. For $1 \leq i \leq l$, let ${}^i\Psi_Y : {}^iW_Y \to Y$ denote the covering morphisms.

Now we can apply proposition 6.3 for each i. So we have an irreducible normal Γ-cover ${}^iW \to T$ satisfying conclusion (1) to (5) of the Proposition 6.3. Also, by Lemma 6.4, we know that the morphism $T \to B$, where B is the locus of $xy - t = 0$ in $D \times_k \mathbb{P}^1_y \times_k \mathrm{Spec}(k[[t]])$, induces an abelian cover of $D \times_k \mathrm{Spec}(k((t)))$. Let V^g denote the generic fiber of a $k[[t]]$-schemes V. Since the branch locus of ${}^iW^g \to T^g$ is determined by the branch locus of ${}^iW \to T$ on the patches. From $(1), (1')$ and (2), we conclude that ${}^iW^g \to T^g$ is ramified only at points of T^g lying above $x = \infty$ since ${}^iW_{YT} \to T_Y$ is étale everywhere and $W_{XT} \to T_X$ is étale away from the points which maps to $D \setminus C = \Theta^{-1}(\{x = \infty\})$ under the compostion of the morphisms $W_{XT} \to T_X \to X \to D$. Also $T^g \to D \times_k \mathrm{Spec}(k((t)))$ is ramified only at points above $D \setminus C = \Theta^{-1}(\{x = \infty\})$, since $T \to B$ is ramified only at points above $x = \infty$ and $y = 0$ and on the generic fiber (i.e., $t \neq 0$) these two points get identified. So for each $1 \leq i \leq l$, we get the following diagram.

$$\begin{tikzcd}
{}^iW^g \arrow[d] \\
{}^iW^g/H \cong W^g_{XT} \arrow[d, "G \text{ cover ramified only at pts lying above } x=\infty"] \\
T^g \arrow[d, "\text{abelian cover ramified only at } x=\infty"] \\
D \times \mathrm{Spec}(k((t)))
\end{tikzcd}$$

T clearly is not defined over k and genus of T^g is least genus of Y, hence at least 1. Now applying Proposition 6.6, we get the following diagram for each i, with same ramification properties as above

$$\begin{tikzcd}
{}^iW^s \arrow[d] \\
{}^iW^s/H \cong W^s_{XT} \arrow[d, "G \text{ cover ramified only at pts lying above } x=\infty"] \\
T^s \arrow[d, "\text{abelian cover ramified only at } x=\infty"] \\
D
\end{tikzcd}$$

where $-^s$, as in Proposition 6.6, denote the specialization to the base field k. Note that $k({}^iW^s)$ are linearly disjoint over $k(W^s_{XT})$ for $1 \leq i \leq l$. So to complete the proof, it is enough to show that for atleast one i, the Galois group of $k({}^iW^s)K^{ab}$ over K^{ab} is Γ, where $k({}^iW^s)$ is the function field of ${}^iW^s$. Note that $k(W_X) \subset k(W^s_{XT}) \subset k(W_X)k(T^s)$. So $k(W^s_{XT})K^{ab} = k(W_X)K^{ab}$, since $k(T^s) \subset K^{ab}$. By assumption Galois group of $k(W_X)K^{ab}$ over K^{ab} is G. So it is enough to show that the Galois group of $k({}^iW^s)K^{ab}$ over $k(W^s_{XT})K^{ab}$ is H for some i. Since H is minimal normal subgroup of Γ, $H \cong \mathbb{S} \times \mathbb{S} \times .. \times \mathbb{S}$, for some simple group $\mathbb{S}$. If $\mathbb{S}$ is non abelian then $\mathbb{S}$ and hence H is perfect. $\mathrm{Gal}(k({}^iW^s)/k(W^s_{XT}))$

is perfect and $k(W^s_{XT})K^{ab}/k(W^s_{XT})$ is a pro-abelian field extension, so they are linearly disjoint. Hence $\mathrm{Gal}(k({}^iW^s)K^{ab}/k(W^s_{XT}K^{ab}) \cong H$. If $\mathbb{S}$ is abelian then $\mathbb{S} \cong \mathbb{Z}/q\mathbb{Z}$ for some prime q different from p. By Grothedieck's result on prime-to-p part of the fundamental group (see Theorem 3.4), there are only finitely many nontrivial surjections from $\pi_1(C)$ to H. These epimorphisms correspond to the H-covers Z_j of D which are étale over C. Now note that we could have chosen l to be any integer. So let l be an integer greater that the number of such H-covers of D. After base change, some of these $Z_j \times_D W^s_{XT}$ may still be H-covers of W^s_{XT}. We choose an i such that ${}^iW^s$ is different from $Z_j \times_D W^s_{XT}$ for all j. For such an i, $k({}^iW^s)$ is linearly disjoint with $k(W^s_{XT})K^{ab}$ over $k(W_{XT})$, since subfields of $k(W^s_{XT})K^{ab}$ which are finite extensions of $k(W^s_{XT})$ are in bijective correspondence with the covers of W^s_{XT} obtained from base change of an étale cover of C. So we found a proper solution to the embedding problem.

□

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